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An Optimum Problem for Linear Systems with Phase Constraints

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1. Our starting point is the following problem [1]. Consider the system

$$\frac{dx}{dt} = Ax + bu$$

$$x \in R^n, \qquad t \in [0, T], \qquad x(0) = x_0, \qquad b \in R^n, \qquad u(t) \in [0, 1], \tag{1}$$

and the functions g and l, piecewise continuous on $[0, T]$. Suppose that $l \geqslant 0$ on $[0, T]$. The control function $u(t)$ is admissible if $\int_0^T l(t)\, u(t) \leqslant \gamma$; an admissible control function is optimal if for the corresponding solution, $\int_0^T g(t)\, x(t)\, dt$ is minimal.

This problem can be transformed in a problem with phase constraints. Let

$$x^0(t) = \int_0^t g(s)\, x(s)\, ds, \qquad x^{n+1}(t) = \int_0^t l(s)\, u(s)\, ds;$$

we get the system

$$\frac{dx^0}{dt} = g(t)\, x, \qquad x^0(0) = 0, \qquad x(0) = x_0, \qquad x^{n+1}(0) = 0,$$

$$\frac{dx}{dt} = Ax + bu, \qquad \frac{dx^{n+1}}{dt} = l(t)\, u. \tag{2}$$

From $l(t) \geqslant 0$ and $u(t) \in [0, 1]$ for $t \in [0, T]$, it follows that the condition $x^{n+1}(T) \leqslant \gamma$ is equivalent to $x^{n+1}(t) \leqslant \gamma$ for $t \in [0, T]$. Thus the control is admissible if $x^{n+1}(t) \leqslant \gamma$ for $t \in [0, T]$ and it is an optimal control if $x_0(T)$ is minimal.

Consider the "adjoint" system

$$\frac{d\psi_0}{dt} = 0, \qquad \psi_0(T) = -1, \qquad \psi(T) = 0, \qquad \psi_{n+1}(T) = k \leqslant 0$$

$$\frac{d\psi}{dt} = -\psi_0 g - \psi A, \qquad \frac{d\psi_{n+1}}{dt} = 0 \tag{3}$$

and the sets

$$E_k{}^+ = \{t \in [0, T]; \psi(t)\, b + kl(t) \geqslant 0\},$$
$$E_k{}^- = \{t \in [0, T]; \psi(t)\, b + kl(t) < 0\}.$$

Let

$$k_0 = \sup \left\{ k \leqslant 0, \int_{E_k+} l(t)\, dt \leqslant \gamma \right\}.$$

Define the control function $\tilde{u}$ equal to 1 on $E_{k_0}^+$ and to 0 on $E_{k_0}^-$.

PROPOSITION 1. *The control $\tilde{u}$ is admissible and optimal.*

PROOF. We have

$$\int_0 l(t)\, \tilde{u}(t)\, dt = \int_{E_{k_0}^+} l(t)\, dt + \int_{E_{k_0}^-} l(t) \cdot 0 \cdot dt = \int_{E_{k_0}^+} l(t)\, dt \leqslant \gamma,$$

hence $\tilde{u}$ is admissible.

If $\int_{E_{k_0}^+} l(t)\, dt = \gamma$ we have $\int_0^T l(t)\, \tilde{u}(t)\, dt = \gamma$; for all other admissible control functions we have $\int_0^T l(t)\, u(t)\, dt \leqslant \gamma$, hence

$$\int_0^T l(t)\,(\tilde{u}(t) - u(t))\, dt \geqslant 0 \qquad \text{and} \qquad k_0 \int_0^T l(t)\,(\tilde{u}(t) - u(t))\, dt \leqslant 0.$$

From the system (3) we get

$$\int_0^T g(x - \tilde{x})\, dt = \int_0^T (\dot{\psi} + \psi A)(x - \tilde{x})\, dt$$
$$= \int_0^T \dot{\psi}(x - \tilde{x})\, dt + \int_0^T \psi A(x - \tilde{x})\, dt.$$

Integrating by parts, we get further

$$\int_0^T \dot{\psi}(x - \tilde{x})\, dt = -\int_0^T \psi A(x - \tilde{x})\, dt + \int_0^T \psi b(\tilde{u} - u)\, dt$$

hence

$$\int_0^T g(x - \tilde{x})\, dt = \int_0^T \psi b(\tilde{u} - u)\, dt.$$

It follows that

$$\int_0^T g(x - \tilde{x})\, dt \geqslant \int_0^T (\psi b + k_0 l)(\tilde{u} - u)\, dt$$

i.e.

$$\int_0^T g(x - \tilde{x})\, dt \geqslant \int_{E_{k_0}^+} [\psi b + k_0 l]\,(1 - u)\, dt + \int_{E_{k_0}^-} [\psi b + k_0 l]\,(-u)\, dt.$$

From the definition of the sets $E^+_{k_0}$ and $E^-_{k_0}$ we deduce that $\int_0^T g(x - \tilde{x})\,dt \geqslant 0$ and the control $\tilde{u}$ is optimal.

If $k_0 < 0$ we have the situation $\int_{E^+_{k_0}} l\,dt = \gamma$; if $k_0 = 0$ we have

$$\int_0^T g(x - \tilde{x})\,dt = \int_0^T \psi b(\tilde{u} - u)\,dt$$

$$= \int_{E^+_{k_0}} \psi b(1 - u)\,dt + \int_{E^-_{k_0}} \psi b(-u)\,dt \geqslant 0$$

hence $\tilde{u}$ is again optimal.

REMARKS. 1. The control $\tilde{u}$ we have obtained is the same as in [1] as one may see if ψ is explicitly written from (3).

2. The optimal control realizes the maximum relative to u, of the function

$$H = \psi_0 g x + \psi(Ax + bu) + \psi_{n+1} l u.$$

3. Suppose that the admissible control functions are defined by the condition $\int_0^{t_1} lu\,dt \leqslant \gamma$, $t_1 < T$. We can choose $l \equiv 0$ on $[t_1, T]$ so that we get again the problem we have considered.

2. The last remark suggests the following new problem. Consider the system (1) and suppose that $l(t) \geqslant 0$ for $t \in [0, t_1]$, $l(t) < 0$ for $t \in (t_1, t_2]$, $l(t) \geqslant 0$ for $t \in (t_2, T]$, $0 < t_1 < t_2 < T$. The control function will be admissible if $\int_0^{t_1} lu\,dt \leqslant \gamma$, $\int_0^T lu\,dt \leqslant \gamma$. It will be optimal if it is admissible and $\int_0^T gx\,dt$ is minimal.

This new problem is also transformed in a problem with phase constraints. By the same notations as above we get the system (2), the control being admissible if $x^{n+1}(t_1) \leqslant \gamma$, $x^{n+1}(T) \leqslant \gamma$, which is equivalent, by the properties of $l(t)$, to $x^{n+1}(t) \leqslant \gamma$ for $t \in [0, T]$.

Consider again the adjoint system (3) and the sets E_k^+, E_k^- defined as above. Write

$$m = \left\{ k \leqslant 0, \int_{E_k^+} l(t)\,dt \leqslant \gamma \right\}$$

The set m is not void; indeed for $0 < \epsilon < \gamma/T$, a $\tilde{k} \leqslant 0$ exists such that $E_{\tilde{k}}^+ \cap E_\epsilon = \phi$ where $E = \{t \in [0, T]; l(t) \geqslant \epsilon\}$, hence

$$\int_{E_{\tilde{k}}^+} l\,dt < \epsilon \mu(E_{\tilde{k}}^+) < \epsilon T < \gamma.$$

Let $k_0 = \sup\{k, k \in m\}$. Denote

$$m_1 = \left\{ k \leqslant k_0,\ \int_{E_k^+ \cap [0, t_1)} l(t)\,dt \leqslant \gamma \right\}.$$

We have $m_1 \neq \phi$ since

$$\int_{E_k^- + \cap [0, t_1)} l(t)\, dt < \epsilon\mu(E_k{}^+) < \gamma.$$

Let

$$k_1 = \sup \{k, k \in m_1\}.$$

Denote

$$m_2 = \left\{k_0 \leqslant k \leqslant 0, \int_{E_{k_1}^+ \cap [0, t_1]} l(t)\, dt + \int_{E_k + \cap (t_1, T]} l(t)\, dt \leqslant \gamma\right\}.$$

We have $m_2 \neq \phi$ since

$$\int_{E_{k_1}^+ \cap [0, t_1]} l(t)\, dt + \int_{E_{k_0}^+ \cap (t_1, T]} l(t)\, dt \leqslant \int_{E_{k_0}^+} l(t)\, dt \leqslant \gamma.$$

Let

$$k_0' = \sup \{k, k \in m_2\}.$$

Consider the solution of the system (3) defined by the conditions

$$\psi_0(T) = -1, \qquad \psi(T) = 0, \qquad \psi_{n+1}(T) = k_0',$$
$$\psi_{n+1}(t_1 + 0) = k_0', \qquad \psi_{n+1}(t_1 - 0) = k_1 .$$

Define the control function $\tilde{u} = 1$ on

$$(E_{k_1}^+ \cap [0, t_1]) \cup (E_{k_0'}^+ \cap (t_1, T])$$

and $\tilde{u} = 0$ on

$$(E_{k_1}^- \cap [0, t_1]) \cup (E_{k_0'}^- \cap (t_1, T]).$$

PROPOSITION 2. *The control function $\tilde{u}$ is admissible and optimal.*

PROOF. For $t \in [0, t_1]$ we have

$$\int_0^t l(s)\, \tilde{u}(s)\, ds \leqslant \int_0^{t_1} l(s)\, \tilde{u}(s)\, ds = \int_{E_{k_1}^+} l(t)\, dt \leqslant \gamma;$$

for $t \in (t_2, T]$ we have

$$\int_0^t l(s)\, \tilde{u}(s)\, ds \leqslant \int_0^T l(s)\, \tilde{u}(s)\, ds = \int^T_{E_{k_1}^+ \cap [0, t_1]} l(t)\, dt + \int_{E_{k'_0}^+ \cap (t_1, T]} l(t)\, dt \leqslant \gamma$$

and for $t \in (t_1, t_2)$ we have

$$\int_0^t l(s)\, \tilde{u}(s)\, ds = \int_0^{t_1} l(s)\, \tilde{u}(s)\, ds + \int_{t_1}^t l(s)\, \tilde{u}(s)\, ds \leqslant \int_0^{t_1} l(s)\, \tilde{u}(s)\, ds \leqslant \gamma,$$

hence $\tilde{u}$ is admissible.

In order to prove that $\tilde{u}$ is optimal we shall use the following general lemma.

Consider the system

$$\frac{dx}{dt} = A(t)\,x + B(t)\,u$$

$$t \in [0, T], \qquad x \in R^n, \qquad x(0) = x_0\,, \qquad u(t) \in U \subset E^r. \tag{4}$$

A control u is admissible if for the corresponding solution we have $bx(t) \leqslant \gamma$ for all $t \in [0, T]$; b is a row vector. A control u is optimal if it is admissible and if, for the corresponding solution, $cx(T)$ is minimal; c is a row vector. The solution $x(t)$ corresponding to an admissible control may have arcs on the boundary $bx = \gamma$, or isolated points on this boundary. The extremities of the intervals in which the solution $x(t)$ verifies the relation $bx(t) = \gamma$ and the isolated points in which this relation is verified will be called contact points. The point $t = T$ may be a contact point.

Let $\tilde{u}$ be a control function with the following properties:

1. $\tilde{u}$ is admissible and the corresponding solution admits a finite number of contact points τ_i, $i = 1, \cdots, s$.

2. There exist a piecewise continuous function $\alpha(t)$ ($\alpha(t) \geqslant 0$ for the values t for which the solution is on the boundary, and $\alpha(t) = 0$ for the values t for which the solution is in the domain $bx < \gamma$), and constants $\mu_i \geqslant 0$ such that if $\tilde{\psi}$ is the solution of the system

$$\frac{d\tilde{\psi}}{dt} = -\tilde{\psi}A(t) + \alpha(t)\,b$$

$$\tilde{\psi}(T + 0) = -c, \qquad \tilde{\psi}(\tau_i + 0) = \tilde{\psi}(\tau_i - 0) + \mu_i b \tag{5}$$

then $\tilde{\psi}(t)\,B(t)\,\tilde{u}(t) \geqslant \tilde{\psi}(t)\,B(t)\,u$ for all $u \in U$, and $t \neq \tau_i$.

LEMMA 1. *A control function $\tilde{u}$ with properties 1 and 2 is optimal.*

PROOF. Let u be an admissible control, x the corresponding solution. We have

$$\int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,[\dot{\tilde{x}}(t) - \dot{x}(t)]\,dt = \tilde{\psi}(\tau_{i+1} - 0)\,[\tilde{x}(\tau_{i+1}) - x(\tau_{i+1})]$$
$$- \tilde{\psi}(\tau_i + 0)\,[\tilde{x}(\tau_i) - x(\tau_i)] - \int_{\tau_i}^{\tau_{i+1}} \dot{\tilde{\psi}}(t)\,[\tilde{x}(t) - x(t)]\,dt.$$

Using the system (5) we get

$$\int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,[\dot{\tilde{x}}(t) - \dot{x}(t)]\,dt = \tilde{\psi}(\tau_{i+1} - 0)\,[\tilde{x}(\tau_{i+1}) - x(\tau_{i+1})]$$

$$- \tilde{\psi}(\tau_i + 0)\,[\tilde{x}(\tau_i) - x(\tau_i)] + \int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,A(t)\,[\tilde{x}(t) - x(t)]\,dt$$

$$- \int_{\tau_i}^{\tau_{i+1}} \alpha(t)\,b[\tilde{x}(t) - x(t)]\,dt.$$

From system (4) we have

$$\int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,[\dot{\tilde{x}}(t) - \dot{x}(t)]\,dt = \int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,A(t)\,[\tilde{x}(t) - x(t)]\,dt$$

$$+ \int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,B(t)\,(\tilde{u} - u)\,dt.$$

Hence

$$\tilde{\psi}(\tau_{i+1} - 0)\,[\tilde{x}(\tau_{i+1}) - x(\tau_{i+1})] - \tilde{\psi}(\tau_i + 0)\,[\tilde{x}(\tau_i) - x(\tau_i)]$$

$$= \int_{\tau_i}^{\tau_{i+1}} \tilde{\psi}(t)\,B(t)\,(\tilde{u} - u)\,dt + \int_{\tau_i}^{\tau_{i+1}} \alpha(t)\,b[\tilde{x}(t) - x(t)]\,dt.$$

By adding these relations we finally get

$$Cx(T) - C\tilde{x}(T) = \int_0^T \tilde{\psi}(t)\,B(t)\,(\tilde{u} - u)\,dt + \int_0^T \alpha(t)\,b(\tilde{x}(t) - x(t))\,dt$$

$$+ \sum_{i=1}^{s} \mu_i b(\tilde{x}(\tau_i) - x(\tau_i)).$$

From the properties of $\alpha(t)$ and μ_i it follows that $Cx(T) - C\tilde{x}(T) \geqslant 0$; hence $\tilde{u}$ is optimal.

For our Proposition 2 we shall have $\alpha \equiv 0$, $\mu_1 = k_0' - k_1$, $\mu_2 = -k_0'$. Then $\psi_{n+1}(T - 0) + \mu_2 = 0$, $\psi_{n+1}(t_1 + 0) = \psi_{n+1}(t_1 - 0) + \mu_1$ and it is easy to see that t_1 is contact point, if $k_1 < k_0$; if $k_0' < 0$ then T is also a contact point. From the properties of $l(t)$ it follows that there are no other contact points. The control function $\tilde{u}$ satisfies the conditions 1 and 2 of the lemma because it gives the maximum of the function H from Remark 2. Thus Proposition 2 is completely proved.

3. In the same way may be considered the general case. Suppose $l(t) \geqslant 0$ on the set $m = I_0 \cup I_1 \cup \cdots \cup I_s$, where I_k are closed intervals, and $l(t) < 0$ in the other points of the interval $[0, T]$. Let τ_k, $k = 0, 1, \cdots, s$

be the right extremities of the intervals I_k. A control function will be admissible if for the corresponding solution we have $\int_0^{\tau_0} lu\,dt \leqslant \gamma$, $\int_0^{\tau_1} lu\,dt \leqslant \gamma$, $\cdots \int_0^{\tau_s} lu\,dt \leqslant \gamma$. The control is optimal if it is admissible and if $Cx(T)$ is minimal. This problem is equivalent to a problem with phase constraints for the system

$$\frac{dx}{dt} = A(t)\,x + b(t)\,u, \qquad x(0) = x_0\,, \qquad x^{n+1}(0) = 0,$$

$$\frac{dx^{n+1}}{dt} = l(t)\,u, \qquad x^{n+1}(t) \leqslant \gamma, \qquad t \in [0, T].$$

The adjoint system will be

$$\frac{d\psi}{dt} = -\,\psi A(t), \qquad \psi(T) = -\,C, \qquad \psi_{n+1}(T) = k \leqslant 0$$

$$\frac{d\psi_{n+1}}{dt} = 0.$$

We consider the sets $E_k^+ \cap [0, \tau_s]$, $E_k^- \cap [0, \tau_s]$ and define

$$k_0 = \sup \left\{ k \leqslant 0 ; \int_{E_k^+ \cap [0,\tau_s]} l(t)\,dt \leqslant \gamma \right\}.$$

Consider further the sets $E_k^+ \cap [0, \tau_{s-1}]$, $E_k^- \cap [0, \tau_{s-1}]$ and define

$$k_1 = \sup \left\{ k \leqslant k_0 ; \int_{E_k^+ \cap [0,\tau_{s-1}]} l(t)\,dt \leqslant \gamma \right\}.$$

Continuing in the same manner, consider finally the sets $E_k^+ \cap [0, \tau_0]$, $E_k^- \cap [0, \tau_0]$ and define

$$k_s = \sup \left\{ k \leqslant k_{s-1} ; \int_{E_k^+ \cap [0,\tau_0]} l(t)\,dt \leqslant \gamma \right\}.$$

After this we start with the sets $E_k^+ \cap [\tau_0\,, \tau_1]$, $E_k^- \cap [\tau_0\,, \tau_1]$ and define

$$k'_{s-1} = \sup \left\{ k_{s-1} \leqslant k \leqslant k_{s-2} ; \int_{E_k^+ \cap [\tau_0,\tau_1]} l(t)\,dt + \int_{E_{k_s}^+ \cap [0,\tau_0]} l(t)\,dt \leqslant \gamma \right\},$$

then

$$k'_{s-2} = \sup \left\{ k_{s-2} \leqslant k \leqslant k_{s-3} ; \int_{E_k^+ \cap [\tau_1,\tau_2]} l(t)\,dt \right.$$
$$\left. + \int_{E_{k_1}^+ \cap [\tau_0,\tau_1]} l(t)\,dt + \int_{E_{k_s}^+ \cap [0,\tau_0]} l(t)\,dt \leqslant \gamma \right\}$$

and finally

$$k_0' = \sup \Big\{ k_0 \leqslant k \leqslant 0, \int_{E^+_{k_s} \cap [0,\tau_0]} l(t)\, dt + \int_{E^+_{k'_{s-1}} \cap [\tau_0,\tau_1]} l(t)\, dt + \cdots + \int_{E_k{}^+ \cap (\tau_{s-1},\tau_s]} l(t)\, dt \leqslant \gamma \Big\}.$$

We consider the solution of the adjoint system defined by the conditions

$$\psi(T) = -C, \qquad \psi_{n+1}(T) = 0, \qquad \psi_{n+1}(\tau_i - 0) = k_{s-i}, \qquad k_s' = k_s,$$
$$\psi_{n+1}(\tau_i + 0) = k'_{s-i-1}(k'_{-1} = 0),$$

and define the control $\tilde{u}$ by $\tilde{u} = 1$ on the set

$$E^+_{k_s} \cap [0, \tau_0] \cup E^+_{k'_{s-1}} \cap (\tau_0, \tau_1] \cup \cdots \cup E^+_{k'_0} \cap (\tau_{s-1}, \tau_s] \cup E_0{}^+ \cap (\tau_s, T]$$

$\tilde{u} = 0$ on the set

$$E^-_{k_s} \cap [0, \tau_0] \cup \cdots \cup E^-_{k_s} \cap (\tau_{s-1}, \tau_s] \cup E_0{}^- \cap (\tau_s, T].$$

PROPOSITION 3. *The control $\tilde{u}$ is admissible and optimal.*

PROOF. For $t \in I_{i+1}$ we have

$$\int_0^t l\tilde{u}\, ds \leqslant \int_0^{\tau_{i+1}} l\tilde{u}\, ds$$

and

$$\int_0^{\tau_{i+1}} l\tilde{u}\, ds = \int_{E_{k_s} \cap [0,\tau_0]} l\, dt + \cdots + \int_{E^+_{k'_{s-i-1}} \cap [\tau_i,\tau_{i+1}]} l\, dt \leqslant \gamma.$$

For $t \in (\tau_i, \tau_{i+1})$ and $t \notin I_{i+1}$ we have $\int_0^t l\tilde{u}\, ds \leqslant \int_0^{\tau_i} l\tilde{u}\, ds$ since on $(\tau_i, t]$ we have $l(s) < 0$; in the same way as above it is seen that $\int_0^{\tau_i} l\tilde{u}\, ds \leqslant \gamma$. For $t \in (\tau_s, T]$ we have $\int_0^t l\tilde{u}\, ds \leqslant \int_0^{\tau_s} l\tilde{u}\, ds \leqslant \gamma$, and we have proved that $\tilde{u}$ is admissible. In order to prove that it is optimal we apply the lemma with $\mu_i = k'_{s-i-1} - k_{s-i}$, $i = 0, 1, 2, \cdots, s$.

REMARK. The problems considered above are particular cases of the general problem considered by R. V. Gamkrelidze [2], and L. Berkovitz [3] but their results cannot be applied in our problems, since the regularity conditions [2] are not satisfied. Nevertheless the condition (2) from the lemma corresponds to the necessary conditions in [2] and [3]. In the necessary conditions of S. S. L. Chang [4] the constants μ_i do not appear. We shall show an example for which no α may be found which satisfies the conditions from the lemma if $\mu_i = 0$. Consider the system

$$\frac{dx^0}{dt} = a_1 x^1 \qquad x^0(0) = 0, \qquad x^1(0) = x_0{}^1, \qquad x^2(0) = 0,$$

$$\frac{dx^1}{dt} = a_2 x^1 + a_3 u \qquad l(t) \geqslant 0, \quad \text{for} \quad t \in [0, t_1], \qquad t_1 < T,$$
$$l(t) < 0, \quad \text{for} \quad t \in (t_1, T]$$

$$\frac{dx^2}{dt} = l(t)\, u \qquad a_1 a_2 a_3 > 0, \qquad a_2 < 0.$$

Suppose that l is continuous and that $l(t) > 0$ on $(t_1 - h, t_1)$, $\int_0^{t_1} l\, dt > \gamma$. A control u is admissible if $x^2(t) \leqslant \gamma$ for $t \in [0, T]$ and is optimal if it is admissible and $x^0(T)$ is minimal. By the theorem of Chang we have to consider the system

$$\frac{d\psi_0}{dt} = 0 \qquad \psi_0(T) = -1, \qquad \psi_1(T) = 0, \qquad \psi_2(T) = 0$$

$$\frac{d\psi_1}{dt} = -\psi_0 a_1 - \psi_1 a_2 \qquad \alpha(t) \geqslant 0 \quad \text{for } t \text{ such that} \quad x^2(t) = \gamma$$

$$\frac{d\psi_2}{dt} = \alpha(t) \qquad \alpha(t) = 0 \quad \text{for } t \text{ such that} \quad x^2(t) < \gamma.$$

Let $H = [\psi_1 a_3 + \psi_2 l(t)]\, u$; we have

$$\psi_1 a_3 = -\frac{a_1 a_3}{a_2}\,[1 - e^{a_2(T-t)}] > 0.$$

For $t \in (t_1, T]$ the condition that H be maximum gives
$u = 1$ on the set

$$\left\{ t \in (t_1, T], \psi_1 a_3 + \left(\int_T^t \alpha(s)\, ds\right) l(t) > 0 \right\} = (t_1, T)$$

since $(\int_T^t \alpha\, ds)\, l(t) > 0$ and $\psi_1 a_3 > 0$ for $t \in (t_1, T)$. It follows that $l(t)\, u < 0$ for $t \in (t_1, T)$ and x^2 will be strictly decreasing on (t_1, T). It follows that $x^2(t) < \gamma$ on (t_1, T); hence $\alpha \equiv 0$ on this interval. For $t \in [0, t_1]$ the condition that H be maximum gives

$u = 1$ on the set

$$\left\{ t \in [0, t_1]; \psi_1 a_3 + \left(\int_{t_1}^t \alpha(s)\, ds\right) l(t) \geqslant 0 \right\}$$

$u = 0$ on the set

$$\left\{ t \in [0, t_1]; \psi_1 a_3 + \left(\int_{t_1}^t \alpha(s)\, ds\right) l(t) < 0 \right\}.$$

Since $\psi_1 a_3 + (\int_{t_1}^{t} \alpha(s)\, ds)\, l(t) > 0$ for $t = t_1$ and this function is continuous we have for all α, $\psi_1 a_3 + (\int_{t_1}^{t_1-\epsilon} \alpha(t) dt) l(t) > 0$ for a convenient $\epsilon > 0$. On $(t_1 - \epsilon, t_1) \cap (t_1 - h, t_1)$ we will have $l(t)\, u = l(t) > 0$; hence $x^2(t)$ will be strictly increasing and if $x^2(t) = \gamma$ for $t \leqslant t_1$, we will have $x^2(t) > \gamma$.

If $\alpha \equiv 0$ the condition that H be maximum gives $u = 1$ on $[0, t_1]$ and since $\int_0^{t_1} l dt > \gamma$ it follows $x^2(t_1) > \gamma$ and the control is not admissible.

4. As a further generalization we way consider the system

$$\frac{dx}{dt} = A(t)\, x + B(t)\, u, \qquad x(0) = x_0\,, \qquad y(0) = 0,$$

$$\frac{dy}{dt} = H(t)\, u, \qquad x \in R^n, \qquad y \in R^m, \qquad u \in R^r, \qquad u_j(t) \in [0, 1].$$

Let b be a row vector such that $bH_i \geqslant 0$, where H_i are the columns of the matrix H. A control u is admissible if $\int_0^T bH(t)\, u(t)\, dt \leqslant \gamma$, and it is optimal if $Cx(T)$ is minimal. This problem is again a problem with phase contraints; a control is admissible if and only if for the corresponding solution $by(t) \leqslant \gamma$.

In order to show how this problem may be solved we take for simplicity $r = 2$.

Consider the adjoint system

$$\frac{d\psi_1}{dt} = -\psi_1 A(t), \qquad \psi_1(T) = -C, \qquad \psi_2(T) = kb, \qquad k \leqslant 0$$

$$\frac{d\psi_2}{dt} = 0$$

and define the sets

$$E_k^{i+} = \{t \in [0, T], \psi_1 B_i(t) + kbH_i(t) \geqslant 0\}_{i=1,2}\,,$$

$$E_k^{i-} = \{t \in [0, T], \psi_1 B_i(t) + kbH_i(t) < 0\}$$

$$E_k^{1+,2-} = E_k^{1+} \cap E_k^{2-}\,, \qquad E_k^{1+,2+} = E_k^{1+} \cap E_k^{2+}\,, \qquad E_k^{2+,1-} = E_k^{2+} \cap E_k^{1-,2}\,,$$

$$E_k^{1-,2-} = E_k^{1-} \cap E_k^{2-}.$$

Let

$$k_0 = \sup \left\{ k \leqslant 0; \int_{E_k^{1+}} bH_1 dt + \int_{E_k^{2+}} bH_2 dt \leqslant \gamma \right\}$$

and define the control $\tilde{u} = (\tilde{u}_1\,, \tilde{u}_2)$ such that $\tilde{u} = (1, 1)$ on $E_{k_0}^{1+,2+}$, $\tilde{u} = (1, 0)$ on $E_{k_0}^{1+,2-}$, $\tilde{u} = (0, 1)$ on $E_{k_0}^{1-,2+}$, and $\tilde{u} = (0, 0)$ on $E_{k_0}^{1-,2-}$.

PROPOSITION 4. *The control $\tilde{u}$ is admissible and optimal.*

PROOF. We have

$$\int_0^t (bH_1\tilde{u}_1 + bH_2\tilde{u}_2)\,ds \leqslant \int_0^T (bH_1\tilde{u}_1 + bH_2\tilde{u}_2)\,ds$$
$$= \int_{E_{k_0}^{1+}} bH_1 dt + \int_{E_{k_0}^{2+}} bH_2 dt \leqslant \gamma,$$

$$E_{k_0}^{1+} = E_{k_0}^{1+,2+} \cup E_{k_0}^{1+,2-} \qquad E_{k_0}^{2+} = E_{k_0}^{2+,1+} \cup E_{k_0}^{2+,1-}$$

hence the control is admissible. It is easy to see that T is the only possible contact point. The control $\tilde{u}$ gives the maximum of $\mathscr{H} = (\psi_1 B + \psi_2 H)$ and $\mu_0 = -k_0 \geqslant 0$ if

$$\int_{E_{k_0}^{1+}} bH_1 dt + \int_{E_k^{2+}} bH_2 dt = \gamma,$$

$\mathscr{H} = 0$ if

$$\int_{E_{k_0}^{1+}} bH_1 dt + \int_{E_{k_0}^{2+}} bH_2 dt < \gamma^-$$

the optimality of $\tilde{u}$ follows from the lemma.

REMARK. If on $[t_1, t_2] \subset [0, T]$ we have $bH_1 \geqslant 0$, $bH_2 < 0$, the above construction will not give an admissible control. In this case it will probably be necessary to introduce a function α as in the lemma but the construction of this function is an open problem.

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